ON NORMAL FORMS FOR LEVI-FLAT HYPERSURFACES WITH AN ISOLATED LINE SINGULARITY

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Abstract. We prove the existence of normal forms for some local real-analytic Levi-flat hypersurfaces with an isolated line singularity. We also give sufficient conditions for that a Levi-flat hypersurface with a complex line as singularity to be a pullback of a real-analytic curve in $\mathbb{C}$ via a holomorphic function.

1. Introduction

A real-analytic hypersurface M (possibly singular) in a complex manifold is said to be *Levi-flat*, if its regular part is locally foliated by complex hypersurfaces. The aim of this paper is to prove the existence of some normal forms for local real-analytic Levi-flat hypersurfaces defined by the vanishing of real part of holomorphic functions with an *isolated line singularity* (for short: ILS). In particular, we establish an analogous result like in Singularity Theory for germs of holomorphic functions.

The main motivation for this work is a result due to Dirk Siersma, who introduced in [14] the class of germs of holomorphic functions with an ILS. More precisely, let $\mathcal{O}_{n+1} := \{f : (\mathbb{C}^{n+1}, 0) \to \mathbb{C}\}$ be the ring of germs of holomorphic functions and let m be its maximal ideal. If $(x, y) = (x, y_1, \ldots, y_n)$ denote the coordinates in $\mathbb{C}^{n+1}$ and consider the line $L := \{y_1 = \ldots = y_n = 0\}$, let $I := (y_1, \ldots, y_n) \subset \mathcal{O}_{n+1}$ be its ideal and denote by $\mathcal{D}_I$ the group of local analytic isomorphisms $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ for which $\varphi(L) = L$. Then $\mathcal{D}_I$ acts on I^2 and for $f \in I^2$, the tangent space of (the orbit of) f with respect to this action is the ideal defined by

$$\tau(f) := m\frac{\partial f}{\partial x} + (y)(\frac{\partial f}{\partial y})$$

and the codimension of (the orbit) of f is

$$c(f) := \dim_{\mathbb{C}} \frac{I^2}{\tau(f)}.$$

A line singularity is a germ $f \in I^2$. An ILS is a line singularity f such that $c(f) < \infty$. Geometrically, $f \in I^2$ is an ILS if and only if the singular locus of f is L and for every $x \neq 0$, the germ of (a representative of) f at $(x, 0) \in L$ is equivalent to $y_1^2 + \ldots + y_n^2$. In a certain sense ILS are the first generalization of isolated singularities. D. Siersma proved the following result. (The topology on $\mathcal{O}_{n+1}$ is introduced as in [5, p. 145]).

Theorem 1.1. *A germ $f \in I^2$ is D_I-simple (i.e. $c(f) < \infty$ and f has a neighborhood in I^2 which intersects only a finite number of D_I-orbits) if and only if f is D_I-equivalent to one the germs in the following table*

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Type	*Normal form*	*Conditions*
A_∞	$y_1^2+y_2^2+\ldots+y_n^2$	
D_∞	$xy_1^2+y_2^2+\ldots+y_n^2$	
$J_{k,\infty}$	$x^ky_1^2+y_1^3+y_2^2+\ldots+y_n^2$	$k\geq 2$
$T_{\infty,k,2}$	$x^2y_1^2+y_1^k+y_2^2+\ldots+y_n^2$	$k\geq 4$
$Z_{k,\infty}$	$xy_1^3+x^{k+2}y_1^2+y_2^2+\ldots+y_n^2$	$k\geq 1$
$W_{1,\infty}$	$x^3y_1^2+y_1^4+y_2^2+\ldots+y_n^2$	
$T_{\infty,q,r}$	$xy_1y_2+y_1^q+y_2^r+y_3^2\ldots+y_n^2$	$q\geq r\geq 3$
$Q_{k,\infty}$	$x^ky_1^2+y_1^3+xy_2^2+y_3^2\ldots+y_n^2$	$k\geq 2$
$S_{1,\infty}$	$x^2y_1^2+y_1^2y_2+y_3^2+\ldots+y_n^2$	

Table 1. Isolated Line singularities

The singularities in Theorem 1.1 are analogous of the A-D-E singularities due to Arnold [1]. A new characterization of simple ILS have been proved by A. Zaharia [15]. We prove the existence of normal forms for Levi-flat hypersurfaces with an ILS.

Theorem 1.2. *Let $M=\{F=0\}$ be a germ of an irreducible real-analytic Levi-flat hypersurface on $(\mathbb{C}^{n+1},0)$, $n\geq 3$. Suppose that*

$$F(x,y)=\mathcal{R}e(P(x,y))+H(x,y),$$

where $P(x,y)$ is one of the germs of the Table 1 and H is a germ of real-analytic function of higher order terms such that $H(x,0)=0$. Then there exists a biholomorphism $\varphi:(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$ preserving L such that

$$\varphi(M)=\{\mathcal{R}e(P(x,y))=0\}.$$

This result is a Siersma's type Theorem for singular Levi-flat hypersurfaces. We remark that if $\varphi(M)=\{\mathcal{R}e(P(x,y))=0\}$, where P is a germ with an ILS at L then $\mathsf{Sing}(M)=L$. In other words, M is a Levi-flat hypersurface with an ILS at L. If $P(x,y)$ is the germ A_∞, we prove that Theorem 1.2 is true in the case $n=2$.

Theorem 1.3. *Let $M=\{F=0\}$ be a germ of an irreducible real-analytic Levi-flat hypersurface on $(\mathbb{C}^3,0)$. Suppose that F is defined by*

$$F(x,y)=\mathcal{R}e(y_1^2+y_2^2)+H(x,y),$$

where H is a germ of real-analytic function such that $H(x,0)=0$ and $j^k(H)=0$ for $k=2$. Then there exists a biholomorphism $\varphi:(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ preserving L such that $\varphi(M)=\{\mathcal{R}e(y_1^2+y_2^2)=0\}$.

The above result should be compared to [2, Theorem 1.1]. This result can be viewed as a Morse's Lemma for Levi-flat hypersurfaces with an ILS at L. The problem of normal forms of Levi-flat hypersurfaces in $\mathbb{C}^3$ with an ILS seems difficult in the other cases. To prove these results we use techniques of holomorphic foliations developed in [4] and [6]. Another normal forms of singular Levi-flat hypersurfaces have been obtained in [2], [7] and [9].

This paper is organized as follows: In Section 2, we recall some definitions and known results about Levi-flat and holomorphic foliations. Section 3 is devoted to prove Theorem 1.2. In Section 4, we prove Theorem 1.3. Finally, in Section 5, using holomorphic foliations, we give sufficient conditions for that a Levi-flat hypersurface with a complex line as singularity to be a pullback of a real-analytic curve in $\mathbb{C}$ via a holomorphic function, (see Theorem 5.7).

2. Levi-flat hypersurfaces and Foliations

Let $M = \{F = 0\}$ be a germ at $0 \in \mathbb{C}^{n+1}$ of an irreducible real-analytic hypersurface. Let us denote by $M^* := \{F = 0\} \backslash \{dF = 0\}$ the regular part of M and by

$$\mathsf{Sing}(M) := \{F = 0\} \cap \{dF = 0\}, \tag{2.1}$$

the singular set of M (or "set of critical points" of M). Note that $\mathsf{Sing}(M)$ contains all points $q \in M$ such that M is smooth at q, but the codimension of M at q is at least two. In general the singular set of a real-analytic subvariety M in a complex manifold is defined as the set of points near which M is not a real-analytic submanifold (of any dimension) and "in general" has structure of a semianalytic set; see for instance, [11]. In this paper, we work with $\mathsf{Sing}(M)$ as defined in (2.1). We recall that the Levi distribution L on M^* is defined by

$$L_p := ker(\partial F(p)) \subset T_p M^* = ker(dF(p)), \quad \text{for any } p \in M^*. \tag{2.2}$$

We say that M is *Levi-flat* if the Levi distribution on M^* is integrable, in Frobenius sense. The integrability condition of L implies that M^* is foliated by complex codimension one holomorphic submanifolds immersed on M^*.

The foliation defined by L is called the *Levi-foliation* and will be denoted by $\mathcal{L}$. Notice that the Levi distribution L on M^* can be defined by the real-analytic 1-form $\eta = i(\partial F - \bar{\partial} F)$, which is called the Levi 1-form of F. It is well known that the integrability condition of L is equivalent to equation $(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F|_{M^*} = 0$.

Let us consider the series Taylor of F at $0 \in \mathbb{C}^{n+1}$,

$$F(x, y) = \sum_{i,\mu,j,\nu} F_{i\mu j\nu} x^i y^\mu \bar{x}^j \bar{y}^\nu$$

where $\bar{F}_{i\mu j\nu} = F_{j\nu i\mu}$; $i, j \in \mathbb{N}$, $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, $(x, y) \in \mathbb{C} \times \mathbb{C}^n$, $y^\mu = y_1^{\mu_1} \ldots y_n^{\mu_n}$ and $\bar{y}^\nu = \bar{y}_1^{\nu_1} \ldots \bar{y}_n^{\nu_n}$. The complexification $F_{\mathbb{C}} \in \mathcal{O}_{2n+2}$ of F is defined by the serie

$$F_{\mathbb{C}}(x, y, z, w) = \sum_{i,\mu,j,\nu} F_{i\mu j\nu} x^i y^\mu z^j w^\nu,$$

where $z \in \mathbb{C}$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and $w^\nu = w_1^{\nu_1} \ldots w_n^{\nu_n}$. Notice that $F(x, y) = F_{\mathbb{C}}(x, y, \bar{x}, \bar{y})$. The complexification $M_{\mathbb{C}}$ of M is defined as $M_{\mathbb{C}} := \{F_{\mathbb{C}} = 0\}$ and defines a complex subvariety in $\mathbb{C}^{2n+2}$, its regular part is $M_{\mathbb{C}}^* := M_{\mathbb{C}} \backslash \{dF_{\mathbb{C}} = 0\}$. Now, assume that M is Levi-flat. Then the integrability condition of

$$\eta = i(\partial F - \bar{\partial} F)|_{M^*}$$

implies that $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ is integrable, where

$$\eta_{\mathbb{C}} := i[(\partial_x F_{\mathbb{C}} + \partial_y F_{\mathbb{C}}) - (\partial_z F_{\mathbb{C}} + \partial_w F_{\mathbb{C}})].$$

Therefore $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ defines a codimension one holomorphic foliation $\mathcal{L}_{\mathbb{C}}$ on $M_{\mathbb{C}}^*$ that will be called the complexification of $\mathcal{L}$.

Let $W := M_{\mathbb{C}}^* \backslash \mathsf{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$ and denote by L_ζ the leaf of $\mathcal{L}_{\mathbb{C}}$ through ζ, where $\zeta \in W$. The next results will be used several times along of the paper.

Lemma 2.1 (Cerveau-Lins Neto [4]). *For any $\zeta \in W$, the leaf L_ζ of $\mathcal{L}_{\mathbb{C}}$ through ζ is closed in $M_{\mathbb{C}}^*$.*

Definition 2.2. The algebraic dimension of $\mathsf{Sing}(M)$ is the complex dimension of the singular set of $M_{\mathbb{C}}$.

The following result will be enunciated in the context of Levi-flat hypersurfaces in $\mathbb{C}^{n+1}$.

Theorem 2.3 (Cerveau-Lins Neto [4]). *Let $M = \{F = 0\}$ be a germ of an irreducible analytic Levi-flat hypersurface at $0 \in \mathbb{C}^{n+1}$, $n \geq 2$, with Levi 1-form $\eta = i(\partial F - \bar{\partial} F)$. Assume that the algebraic dimension of $\mathsf{Sing}(M) \leq 2n-2$. Then there exists a unique germ at $0 \in \mathbb{C}^{n+1}$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, if one of the following conditions is fulfilled:*

(1) *$n \geq 3$ and $cod_{M^*_{\mathbb{C}}}(\mathsf{Sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 3$.*
(2) *$n \geq 2$, $cod_{M^*_{\mathbb{C}}}(\mathsf{Sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 2$ and $\mathcal{L}_{\mathbb{C}}$ admits a non-constant holomorphic first integral.*

Moreover, in both cases the foliation $\mathcal{F}_M$ admits a non-constant holomorphic first integral f such that $M = \{\mathcal{R}e(f) = 0\}$.

3. Proof of Theorem 1.2

We write
$$F(x,y) = \mathcal{R}e(P(x,y_1,\ldots,y_n)) + H(x,y_1,\ldots,y_n),$$
where $P(x,y_1,\ldots,y_n)$ is one of the polynomials of the Table 1 and $H : (\mathbb{C}^{n+1},0) \to (\mathbb{R},0)$ is a germ of real-analytic function of higher order terms such that $H(x,0) = 0$ for all $x \in \mathbb{C}$. The complexification of F given by
$$F_{\mathbb{C}}(x,y,z,w) = \frac{1}{2}P(x,y) + \frac{1}{2}P(z,w) + H_{\mathbb{C}}(x,y,z,w),$$
thus $M_{\mathbb{C}} = \{F_{\mathbb{C}}(x,y,z,w) = 0\} \subset (\mathbb{C}^{2n+2},0)$, where $z \in \mathbb{C}$ and $w = (w_1,\ldots,w_n) \in \mathbb{C}^n$.

Since $P(x,y)$ has an ILS at L, we get $\mathsf{Sing}(M_{\mathbb{C}}) = \{y = w = 0\} \simeq \mathbb{C}^2$. In particular, the algebraic dimension of $\mathsf{Sing}(M)$ is 2. On the other hand, the complexification of $\eta = i(\partial F - \bar{\partial} F)$ is
$$\eta_{\mathbb{C}} := i[(\partial_x F_{\mathbb{C}} + \partial_y F_{\mathbb{C}}) - (\partial_z F_{\mathbb{C}} + \partial_w F_{\mathbb{C}})].$$

Recall that $\eta|_{M^*}$ and $\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$ respectively. Now we compute $\mathsf{Sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$. We can write $dF_{\mathbb{C}} = \alpha + \beta$, with
$$\alpha := \frac{\partial F_{\mathbb{C}}}{\partial x}dx + \sum_{j=1}^{n}\frac{\partial F_{\mathbb{C}}}{\partial y_j}dy_j = \frac{1}{2}\frac{\partial P}{\partial x}(x,y)dx + \frac{1}{2}\sum_{j=1}^{n}\frac{\partial P}{\partial y_j}(x,y)dy_j + \theta_1$$
and
$$\beta := \frac{\partial F_{\mathbb{C}}}{\partial z}dz + \sum_{j=1}^{n}\frac{\partial F_{\mathbb{C}}}{\partial w_j}dw_j = \frac{1}{2}\frac{\partial P}{\partial z}(z,w)dz + \frac{1}{2}\sum_{j=1}^{n}\frac{\partial P}{\partial w_j}(z,w)dw_j + \theta_2$$
where $\theta_1 = \frac{\partial H_{\mathbb{C}}}{\partial x}dx + \sum_{j=1}^{n}\frac{\partial H_{\mathbb{C}}}{\partial z_j}dz_j$ and $\theta_2 = \frac{\partial H_{\mathbb{C}}}{\partial z}dz + \sum_{j=1}^{n}\frac{\partial H_{\mathbb{C}}}{\partial w_j}dw_j$.

Note that $\eta_{\mathbb{C}} = i(\alpha - \beta)$, and so
$$\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = (\eta_{\mathbb{C}} + idF_{\mathbb{C}})|_{M^*_{\mathbb{C}}} = 2i\alpha|_{M^*_{\mathbb{C}}} = -2i\beta|_{M^*_{\mathbb{C}}}. \tag{3.1}$$

In particular, $\alpha|_{M^*_{\mathbb{C}}}$ and $\beta|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}_{\mathbb{C}}$. Therefore $\mathsf{Sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$ can be splited in two parts. In fact, let $M_1 := \{(x,y,z,w) \in M_{\mathbb{C}} | \frac{\partial F_{\mathbb{C}}}{\partial z} \neq 0$ or $\frac{\partial F_{\mathbb{C}}}{\partial w_j} \neq 0$ for some $j = 1,\ldots,n\}$ and $M_2 := \{(x,y,z,w) \in M_{\mathbb{C}} | \frac{\partial F_{\mathbb{C}}}{\partial x} \neq 0$ or $\frac{\partial F_{\mathbb{C}}}{\partial z_j} \neq 0$ for some $j = 1,\ldots,n\}$, then $M_{\mathbb{C}} = M_1 \cup M_2$. If we denote by $A_0 = \frac{\partial H_{\mathbb{C}}}{\partial x}$, $A_j = \frac{\partial H_{\mathbb{C}}}{\partial z_j}$ for all $1 \leq j \leq n$ and by $B_0 = \frac{\partial H_{\mathbb{C}}}{\partial z}$, $B_j = \frac{\partial H_{\mathbb{C}}}{\partial w_j}$ for all $1 \leq j \leq n$, we obtain that $\mathsf{Sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = X_1 \cup X_2$, where
$$X_1 := M_1 \cap \{\frac{\partial P}{\partial x}(x,y) + A_0 = \frac{\partial P}{\partial y_1}(x,y) + A_1 = \ldots = \frac{\partial P}{\partial y_n}(x,y) + A_n = 0\}$$

and

$$X_2 := M_2 \cap \{\frac{\partial P}{\partial z}(z,w) + B_0 = \frac{\partial P}{\partial w_1}(z,w) + B_1 = \ldots = \frac{\partial P}{\partial w_n}(z,w) + B_n = 0\}.$$

Since P is a polynomial with an ILS at $L = \{y = 0\}$, we conclude that

$$cod_{M^*_{\mathbb{C}}} \mathsf{Sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = n.$$

By hypothesis $n \geq 3$, then it follows from Theorem 2.3, part (1) that there exists a germ $f \in \mathcal{O}_{n+1}$ such that the holomorphic foliation $\mathcal{F}$ defined by $df = 0$ is tangent to M. Moreover $M = \{\mathcal{R}e(f) = 0\}$. Note that if $M = \{\mathcal{R}e(f) = 0\} = \{F = 0\}$, with F an irreducible germ, we must have that $\mathcal{R}e(f) = UF$, where U is a germ of real-analytic function with $U(0) \neq 0$. Without lost of generality, we can assume that $U(0) = 1$. In particular, $\mathcal{R}e(f) = UF$ implies that $f = P + h.o.t$. According to Theorem 1.1, there exists a biholomorphism $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ preserving L such that $f \circ \varphi^{-1} = P$, (f is D_I-equivalent to P, because f is a germ with ILS at L). Therefore, $\varphi(M) = \{\mathcal{R}e(P) = 0\}$ and the proof ends.

4. Proof of Theorem 1.3

The idea is to use Theorem 2.3, part (2). In order to prove our result in the case $n = 2$, we are going to prove that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.

We begin by a blow-up along $C := \{y_1 = y_2 = w_1 = w_2 = 0\} \simeq \mathbb{C}^2 \subset \mathbb{C}^6$. Let $F(x, y_1, y_2) = \mathcal{R}e(y_1^2 + y_2^2) + h.o.t$ and $M = \{F = 0\}$ Levi-flat. Its complexification can be written as

$$F_{\mathbb{C}}(x, y_1, y_2, z, w_1, w_2) = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(w_1^2 + w_2^2) + H_{\mathbb{C}}(x, y_1, y_2, z, w_1, w_2).$$

Note that

$$\mathsf{Sing}(M_{\mathbb{C}}) = \{y = w = 0\} = C.$$

Let E be the exceptional divisor of the blow-up $\pi : \tilde{\mathbb{C}}^6 \to \mathbb{C}^6$ along C and denote by $\tilde{M}_{\mathbb{C}} := \overline{\pi^{-1}(M_{\mathbb{C}} \setminus \{C\})} \subset \tilde{\mathbb{C}}^6$ the strict transform of $M_{\mathbb{C}}$ via π and by $\tilde{\mathcal{F}} := \pi^*(\mathcal{L}_{\mathbb{C}})$ the foliation on $\tilde{M}_{\mathbb{C}}$.

Now, we consider an especial situation. Suppose that $\tilde{M}_{\mathbb{C}}$ is smooth and set $\tilde{C} := \tilde{M}_{\mathbb{C}} \cap E$. Moreover, assume that $\tilde{C}$ is invariant by $\tilde{\mathcal{F}}$. Take $S = \tilde{C} \setminus \mathsf{Sing}\tilde{\mathcal{F}}$, then S is a smooth leaf of $\tilde{\mathcal{F}}$. Pick $p_0 \in S$ and a transverse section $\sum$ through p_0. Let $G \subset \mathrm{Diff}(\sum, p_0)$ be the holonomy group of the leaf S of $\tilde{\mathcal{F}}$. Since $\mathsf{dim}(\sum) = 1$, we can assume that $G \subset \mathrm{Diff}(\sum, 0)$. We state a fundamental lemma.

Lemma 4.1 (Fernández-Pérez [9]). *In the above situation, suppose that the following properties are verified:*

1. *For any $p \in S \backslash \mathsf{Sing}(\tilde{\mathcal{F}})$ the leaf L_p of $\tilde{\mathcal{F}}$ through p is closed in S.*
2. *$g'(0)$ is a primitive root of unity, for all $g \in G \backslash \{id\}$.*

Then $\mathcal{L}_{\mathbb{C}}$ admits a non-constant holomorphic first integral.

Proof. Let $G' = \{g'(0)/g \in G\}$ and consider the homomorphism $\phi : G \to G'$ defined by $\phi(g) = g'(0)$. We claim that ϕ is injective. In fact, assume that $\phi(g) = 1$ and suppose by contradiction that $g \neq id$. In this case $g(z) = z + az^{r+1} + \ldots$, where $a \neq 0$. According to [12], the pseudo-orbits of this transformation accumulate at $0 \in (\sum, 0)$, contradicting the fact that the leaves of $\tilde{\mathcal{F}}$ are closed and so the assertion is proved. Now, it suffices to prove that any element $g \in G$ has finite order (cf. [13]). In fact, $\phi(g) = g'(0)$ is a root of unity thus g has finite order because ϕ is injective. Hence, all transformations of G have finite order and G is linearizable.

This implies that there is a coordinate system w on $(\sum, 0)$ such that $G = \langle w \to \lambda w \rangle$, where λ is a d^{th}-primitive root of unity (cf. [13]). In particular, $\psi(w) = w^d$ is a first integral of G, that is $\psi \circ g = \psi$ for any $g \in G$.

Let Γ be the union of the separatrices of $\mathcal{L}_{\mathbb{C}}$ through $0 \in \mathbb{C}^6$ and $\tilde{\Gamma}$ be its strict transform under π. The first integral ψ can be extended to a first integral $\varphi : \tilde{M}_{\mathbb{C}} \backslash \tilde{\Gamma} \to \mathbb{C}$ by setting

$$\varphi(q) = \psi(\tilde{L}_q \cap \sum),$$

where $\tilde{L}_p$ denotes the leaf of $\tilde{\mathcal{F}}$ through q. Since ψ is bounded (in a compact neighborhood of $0 \in \sum$), so is φ. It follows from Riemann extension theorem that φ can be extended holomorphically to $\tilde{\Gamma}$ with $\varphi(\tilde{\Gamma}) = 0$. This provides the first integral of $\mathcal{L}_{\mathbb{C}}$. □

The rest of paper is devoted to prove that we are indeed in the conditions of Lemma 4.1. It is follows from Lemma 2.1 that the leaves of $\mathcal{L}_{\mathbb{C}}$ are closed. Therefore, we need to prove that each generator of the holonomy group G of $\tilde{\mathcal{F}}$ with respect to S has finite order.

Consider for instance the chart $(U_1, (x, t, s, z, u, v))$ of $\tilde{\mathbb{C}}^6$ where

$$\pi(x, t, s, z, u, v) = (x, tu, su, z, u, vu) = (x, y_1, y_2, z, w_1, w_2).$$

We have

$$\tilde{M}_{\mathbb{C}} \cap U_1 = \{(x, t, s, z, u, v) \in U_1 | 1 + t^2 + s^2 + v^2 + uH_1(x, t, s, z, u, v) = 0\},$$

where $H_1 = H(x, ut, us, z, u, uv)/u^3$ and this fact imply that

$$E \cap \tilde{M}_{\mathbb{C}} \cap U_1 = \{(x, t, s, z, u, v) \in U_1 | 1 + t^2 + s^2 + v^2 = u = 0\}.$$

It is not difficult to see that these complex subvarieties are smooth. Now, let us describe the foliation $\tilde{\mathcal{F}}$ on U_1. In fact, note that the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M^*_{\mathbb{C}}} = 0$, where

$$\alpha = \frac{1}{2}\frac{\partial P}{\partial x}dx + \frac{1}{2}\frac{\partial P}{\partial y_1}dy_1 + \frac{1}{2}\frac{\partial P}{\partial y_2}dy_2 + \frac{\partial H_{\mathbb{C}}}{\partial x}dx + \sum_{j=1}^{2}\frac{\partial H_{\mathbb{C}}}{\partial y_j}dy_j.$$

It follows that $\alpha = y_1 dy_1 + y_2 dy_2 + \frac{\partial H_{\mathbb{C}}}{\partial x}dx + \sum_{j=1}^{2}\frac{\partial H_{\mathbb{C}}}{\partial y_j}dy_j$, then $\tilde{\mathcal{F}}|_{U_1}$ is defined by $\tilde{\alpha}|_{\tilde{M}_{\mathbb{C}} \cap U_1} = 0$, where

$$\tilde{\alpha} = (t^2 + s^2)du + utdt + usds + u\tilde{\theta}, \tag{4.1}$$

and

$$\tilde{\theta} = \frac{\pi^*(\frac{\partial H_{\mathbb{C}}}{\partial x}dx + \sum_{j=1}^{2}\frac{\partial H_{\mathbb{C}}}{\partial y_j}dy_j)}{u^2}.$$

Therefore, the singular set of $\tilde{\mathcal{F}}|_{U_1}$ is given by

$$\mathsf{Sing}\tilde{\mathcal{F}}|_{U_1} = \{u = t + is = 0\} \cup \{u = t - is = 0\}.$$

On the other hand, note that the exceptional divisor E is invariant by $\tilde{\mathcal{F}}$ and the intersection with $\mathsf{Sing}\widetilde{\mathcal{F}}$ is

$$\mathsf{Sing}\tilde{\mathcal{F}}|_{U_1} \cap E = \{u = t + is = v^2 + 1 = 0\} \cup \{u = t - is = v^2 + 1 = 0\}.$$

In particular, $S := [E \cap \tilde{M}_{\mathbb{C}}] \backslash \mathsf{Sing}\widetilde{\mathcal{L}}_{\mathbb{C}}$ is a leaf of $\tilde{\mathcal{F}}$. We calculate the generators of the holonomy group G of the leaf S. We work in the chart U_1, because of the symmetry of the variables in the definition of the variety $\tilde{M}_{\mathbb{C}}$.

Pick $p_0 = (0, 1, 0, 0, 0, 0) \in S \cap U_1$ and a transversal $\sum = \{(0, 1, 0, 0, \lambda, 0) | \lambda \in \mathbb{C}\}$ parameterized by λ at p_0. We have that

$$\mathsf{Sing}\tilde{\mathcal{F}}|_{U_1} \cap E = \{u = t + is = v^2 + 1 = 0\} \cup \{u = t - is = v^2 + 1 = 0\}.$$

For each $j = 1, 2$; let ρ_j be a 2^{td}-primitive root of -1. The fundamental group $\pi_1(S, p_0)$ can be written in terms of generators as

$$\pi_1(S, p_0) = \langle \gamma_j, \delta_j \rangle_{1 \leq j \leq 2},$$

where for each $j = 1, 2$; γ_j are loops that turn around $\{u = t+is = v-\rho_j = 0\}$ and δ_j are loops that turns around $\{u = t-is = v-\rho_j = 0\}$. Therefore, $G = \langle f_j, g_j\rangle_{1\le j\le 2}$, where f_j and g_j correspond to $[\gamma_j]$ and $[\delta_j]$, respectively. We get from (4.1) that $f_j'(0) = e^{-\pi i}$ and $g_j'(0) = e^{-\pi i}$ for all $1 \le j \le 2$. The proof of the theorem is complete.

5. Levi-flat hypersurfaces with a complex line as singularity

In this section, we work with the system of coordinates $(z_1, \dots, z_n) \in \mathbb{C}^n$. The canonical local models examples of Levi-flat hypersurfaces M in $\mathbb{C}^3$ such that $\mathsf{Sing}(M) = L = \{z_1 = z_2 = 0\}$ are $\{\mathcal{R}e(z_1^2 + z_2^2) = 0\}$ and $\{z_1\bar{z}_2 - \bar{z}_1 z_2 = 0\}$.

Recently, Burns and Gong [2] classified, up to local biholomorphism, all germs of quadratic Levi-flat hypersurfaces. Namely, up to biholomorphism, there is only five models:

Type	Normal form	Singular set
$Q_{0,2k}$	$\mathcal{R}e(z_1^2 + z_2^2 + \ldots + z_k^2)$	$\mathbb{C}^{n-k}$
$Q_{1,1}$	$z_1^2 + 2z_1^2\bar{z}_1 + z_1^2$	empty
$Q_{1,2}^{\lambda}$	$z_1^2 + 2\lambda z_1^2\bar{z}_1 + z_1^2$	$\mathbb{C}^{n-1}$
$Q_{2,2}$	$(z_1 + \bar{z}_1)(z_2 + \bar{z}_2)$	$\mathbb{R}^2 \times \mathbb{C}^{n-2}$
$Q_{2,4}$	$z_1\bar{z}_2 - \bar{z}_1 z_2$	$\mathbb{C}^{n-2}$

Table 2. Levi-flat quadrics

We address the problem of provide conditions to characterise singular Levi-flat hypersurfaces with a complex line as singularity. Using the classification due to Burns and Gong [2], it is not hard to prove the following proposition.

Proposition 5.1. *Suppose that M is a quadratic real-analytic Levi-flat hypersurface in $\mathbb{C}^n$, $n \ge 3$ such that $\mathsf{Sing}(M) = \{z_1 = z_2 = \ldots = z_{n-1} = 0\}$. Then*
1. *If $n = 3$, M is biholomorphically equivalent to $Q_{0,2}$ or $Q_{2,4}$.*
2. *If $n \ge 4$, M is biholomorphically equivalent to $Q_{0,2(n-1)}$.*

Proof. To prove part (1), observe that only there are two models of M which admits $\mathsf{Sing}(M) = \{z_1 = z_2 = 0\}$ as singularity, $Q_{0,2}$ or $Q_{2,4}$. Now to prove part (2), note that if $n \ge 4$, the real hypersurface $\{z_1\bar{z}_2 - \bar{z}_1 z_2 = 0\}$ has a complex subvariety of dimension $n-2$ as singularity. It is follows that M is biholomorphically equivalent to $Q_{0,2(n-1)}$. □

In order to obtain a characterization, we define the Segre varieties associated to real-analytic hypersurfaces. Let M be a real-analytic hypersurface defined by $\{F = 0\}$. Fix $p \in M$, the Segre variety associated to M at p is the complex variety in $(\mathbb{C}^n, p)$ defined by

$$Q_p := \{z \in (\mathbb{C}^n, p) : F_{\mathbb{C}}(z, \bar{p}) = 0\}. \tag{5.1}$$

Now assume that M is Levi-flat and denote by L_p the leaf of $\mathcal{L}$ through $p \in M^*$. We denote by Q_p' the union of all branches of Q_p which are contained in M. Observe that Q_p' could be the empty set when $p \in \mathsf{Sing}(M)$. Otherwise, it is a complex variety of pure dimension $n-1$.

The following result is classical, we proved it here for completeness.

Proposition 5.2. *In above situation, L_p is an irreducible component of (Q_p, p) and $Q_p' = L_p$.*

Proof. Since $p \in M^*$, E. Cartan's theorem assures that there exists a holomorphic coordinate system such that near of p, M is given by $\{\mathcal{R}e(z_n) = 0\}$ and p is the origin. In this coordinates system the foliation $\mathcal{L}$ is defined by $dz_n|_{M^*} = 0$. In particular, $L_0 = \{z_n = 0\}$ and obviously $\{z_n = 0\}$ is a branch of Q_0. Furthermore,

L_0 is the unique germ of complex variety of pure dimension $n-1$ at 0 which is contained in M. Hence $Q'_0 = L_0$. □

Let $p \in \mathsf{Sing}(M)$, we say that p is a *Segre degenerate singularity* if Q_p has dimension n, that is, $Q_p = (\mathbb{C}^n, p)$. Otherwise, we say that p is a *Segre nondegenerate singularity.*

Suppose that M is defined by $\{F = 0\}$ in a neighborhood of p, observe that p is a degenerate singularity of M if $z \longmapsto F_{\mathbb{C}}(z, \bar{p})$ is identically zero.

Remark 5.3. If V is a germ of complex variety of dimension $n-1$ contained in M then for $p \in V$, we have $(V, p) \subset (Q_p, p)$. In particular, if there exists distinct infinitely many complex varieties of dimension $n-1$ through $p \in M$ then p is a Segre degenerate singularity.

To continuation, we consider a germ at $0 \in \mathbb{C}^n$ of a codimension one singular holomorphic foliation $\mathcal{F}$.

Definition 5.4. We say that $\mathcal{F}$ and M are tangent, if the leaves of the Levi foliation $\mathcal{L}$ on M are also leaves of $\mathcal{F}$.

Definition 5.5. A meromorphic (holomorphic) function h is called a meromorphic (holomorphic) first integral for $\mathcal{F}$ if its indeterminacy (zeros) set is contained in $\mathsf{Sing}(\mathcal{F})$ and its level hypersurfaces contain the leaves of $\mathcal{F}$.

Recently, Cerveau and Lins Neto proved the following result.

Theorem 5.6 (Cerveau-Lins Neto [4])**.** *Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of holomorphic codimension one foliation tangent to a germ of an irreducible real analytic hypersurface M. Then $\mathcal{F}$ has a non-constant meromorphic first integral.*

In our context, we prove the following result.

Theorem 5.7. *Let M be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$ of an irreducible real-analytic Levi-flat hypersurfaces such that $\mathsf{Sing}(M) = L := \{z_1 = z_2 = \ldots = z_{n-1} = 0\}$. Suppose that:*

1. *Every point in $\mathsf{Sing}(M)$ is a Segre nondegenerate singularity.*
2. *The Levi-foliation $\mathcal{L}$ on M^* extends to a holomorphic foliation $\mathcal{F}$ in some neighborhood of M.*

Then there exists $f \in \mathcal{O}_n$ and a real-analytic curve $\gamma \subset \mathbb{C}$ such that $M = f^{-1}(\gamma)$.

Proof. Since the Levi-foliation $\mathcal{L}$ on M^* extends to a holomorphic foliation $\mathcal{F}$, we can apply directly Theorem 5.6, this means that $\mathcal{F}$ has a non-constant meromorphic first integral $f = g/h$, where g and h are relatively prime. We asserts that f is holomorphic. In fact, if f is purely meromorphic, we have that for all $\zeta \in \mathbb{C}$, the complex hypersurfaces $V_\zeta = \{g(z) - \zeta h(z) = 0\}$ contains leaves of $\mathcal{F}$. In particular, M contains an infinitely many of hypersurfaces V_ζ, because M is closed and $\mathcal{F}$ is tangent to M. Set $\Lambda := \{\zeta \in \mathbb{C} : V_\zeta \subset M\}$. Note also that the foliation $\mathcal{F}$ is singular at L, so that $\mathcal{I}_f := \{h = g = 0\}$ the indeterminacy set of f intersect L. Therefore, we have a point q at $\mathcal{I}_f \cap L$ which would be a Segre degenerate singularity, because $q \in V_\zeta$, for all $\zeta \in \Lambda$. It is a contradiction and the assertion is proved.

The foliation $\mathcal{F}$ is defined by $df = 0$, $f \in \mathcal{O}_n$ and is tangent to M. Without lost generality, we can assume that f is an irreducible germ in $\mathcal{O}_n$. According to a remark of Brunella [3, pg. 8], there exists a real-analytic curve $\gamma \subset \mathbb{C}$ through the origin such that $M = f^{-1}(\gamma)$. □

Remark 5.8. In [11], J. Lebl gave conditions for the Levi-foliation on M^* does extended to a holomorphic foliation. One could be considered these hypothesis and

establish a theorem more refined. Note also that if $\mathsf{Sing}(M)$ is a germ of smooth complex curve, it is possible adapted the proof of Theorem 5.7. In general, the holomorphic extension problem for the Levi-foliation of a Levi-flat real-analytic hypersurface remains open and is of independent interest, for more details see [8].

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